ISyE 3770
Chapter 2: Probability

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Random Experiments

- An experiment is any action or process whose outcome is subject to uncertainty. A random experiment results in different outcomes, even through it is repeated in the same manner every time.
  - *An experiment may be tossing a coin once or several times, selecting a card from a deck of cards, commuting time from home to work.*
- The set of all possible outcome of a random experiment is called a **sample space**, and is generally denoted as $\mathcal{S}$.
  - *$\mathcal{S}$ is discrete if it consists of a finite or countable infinite set of outcomes.* $\{0, 1\}$ $\{0, 1, 2, \ldots, n\}$
  - *$\mathcal{S}$ is continuous if it contains an interval (either a finite or infinite width) of real numbers.*

$[2, 3]$ $\{[1, 2] \cup [3, 4]\}$

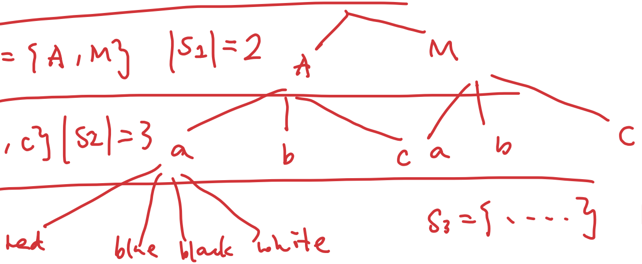
$\{1, 2, [3, 4]\}$

Example

- An order for a car can specify an automatic or manual transmission, three choices of stereo sound, and with any one of the four colors red, blue, black, or white. Describe the set of possible orders for this experiment.

$S_1 = \{A, M\}$ $|S_1| = 2$

$S_2 = \{a, b, c\}$ $|S_2| = 3$



$S_3 = \{\ldots\}$ $|S_3| = 4$

$|S| = |S_1|\,|S_2|\,|S_3| = 2 \times 3 \times 4 = 24$

Sample Space and Probability

- Rolling a dice, there are 6 possible outcomes: $1, 2, \ldots, 6$. If we roll the dice for 2 times, the sample space is:

$$\mathcal{S} = \{(1,1),(1,2),\ldots,(1,6),(2,1),\ldots,(2,6),\ldots,(6,6)\}.$$

- There are 36 different outcomes in the sample space. For a fair dice all 36 outcomes have equal probability or likelihood. So

$$p_1 = \ldots = p_{36} = p.$$

- Since the probability values sum to one, the common probability value is $p = \frac{1}{36}$

Probability of an Event

- An **event** is any collection (subset) of outcomes contained in the sample space $\mathcal{S}$.
- We can also define the probability of an event A, which we denote by $P(A)$.
- The probability $P(A)$ is the sum of the probability values of all the outcomes in the event A.

Probability of an Event

- When an experiment is performed, a particular event A is said to occur if the resulting experimental outcome is contained in A
- **Example:** If we roll a dice twice, we may be interested only in the outcomes when '6' occurs:

$$A = \{(1,6),(2,6),\ldots,(6,6),(6,1),(6,2),\ldots,(6,5)\}$$

which consists of 11 outcomes in the sample space. So the probability of this event is:

$$P(A) = P((1,6)) + \ldots + P((6,6)) + P((6,1)) + \ldots + P((6,5)) = 11\left(\frac{1}{36}\right)$$

Events Combinations

- **Union** of two events is the event consisting of all outcomes that are contained in either of two events, $E_1 \cup E_2$. Called E_1 or E_2.
- **Intersection** of two events is the event consisting of all outcomes that contained in both of two events, $E_1 \cap E_2$. Called E_1 and E_2.
- **Complement** of an event is the set of outcomes that are not contained in the event, E' or not E.

Relations from Set Theory

We can define operations on events based on set theory.

If A and B are two events:

- $A \cup B$ (A or B) is called union
- $A \cap B$ (A and B) is called intersection
- A' (not A) is called complement of A
- $A \setminus B$ (A but not B) is called difference
- If $A \cap B = \emptyset$, then A and B are called *mutually exclusive*.

$A_1\ A_2 \cdots A_n$ $\forall\, i, j$ $A_i \cap A_j = \Phi$ $\Rightarrow A_1 \cdots A_n$ m.e.

Since $A \cap B$, $A \cup B$, A' and $A \setminus B$ are also events, we assign probability values: $P(A \cap B)$, $P(A \cup B)$, $P(A')$ and $P(A \setminus B)$.

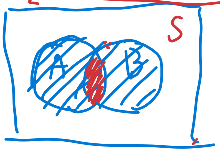
Probability Axioms

The following are important probability axioms.

- $P(A \cap B) = P(A) + P(B) - P(A \cup B)$
- $P(A') = 1 - P(A)$
- $P(\mathcal{S}) = 1$ and $P(\emptyset) = 0$
- Generally, $P(A \setminus B) = P(A) - P(A \cap B)$
    - If $A \cup B = A$ then $P(A \setminus B) = P(A) - P(B)$
- $0 \leq P(A) \leq 1$ for any event A
- It follows that when $A_1, \ldots, A_n$ are n mutually exclusive events then:

$$P(A_1 \cup \ldots \cup A_n) = P(A_1) + \ldots + P(A_n).$$

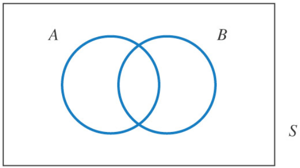
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$



$P(A \cap B) = P(B)$

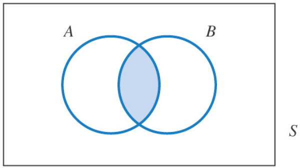
$A \cap B = B$

Probability Axioms



(a)

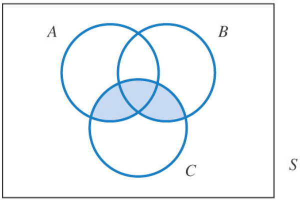
$A \cap B$



(b)

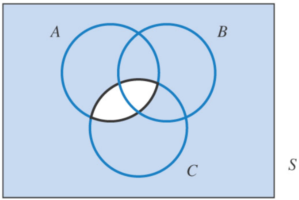
Sample space S with events A and B

$(A \cup B) \cap C$



(c)

$(A \cap C)'$



(d)

Event Relations Laws

- Transitive law (event order is unimportant):
  - $(A \cap B) = (B \cap A)$ and $(A \cup B) = (B \cup A)$
- Distributive law (like in algebra):

$(A+B)\cdot C = A\cdot C + B\cdot C$

  - $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
  - $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

- DeMorgan's laws
  - $(A \cup B)' = A' \cap B'$ the complement of the union is the intersection of the complements
  - $(A \cap B)' = A' \cup B'$ the complement of the intersection is the union of the complements

Example

At Georgia Tech, two major credit card providers want to investigate how many students have a Visa credit card and how many students have a MasterCard.

From our survey we find that the proportion of GT students holding a Visa credit card is 50%, the proportion of GT students holding a MasterCard is 40% and that the proportion of students holding both credit cards is 25%.

Example

Let A be the event that a selected student has a Visa credit card and let B be the event that a selected student has a MasterCard.

Therefore, $P(A) = .5$, $P(B) = .4$ and $P(A \cap B) = .25$.

- What is the probability that the selected individual has at least one of the two types of cards?

  $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.5 + 0.4 - 0.25$
  $= 0.65$

- What is the probability that the selected student has neither type of card?

  $P([A \cup B]') = 1 - 0.65 = 0.35$

- What is the probability that the selected student has a Visa card but not a MasterCard?

Counting Techniques

- In complicated experiments, determining the outcomes in the sample space (or an event) becomes very difficult.
- These are three special rules, or counting techniques, used to determine the number of outcomes in the events and the sample space.
  - Multiplication rule
  - Permutation rule
  - Combination rule
- Each has its special purpose that must be applied properly– the right tool for the right job.

Counting: Multiplication Rule

- Multiplication Rule
  - Let an operation consist of k steps and
    - n_1 ways of completing step 1,
    - n_2 ways of completing step 2, ... and
    - n_k ways of completing step k.
  - Then, the total number of ways or outcomes are:

$$n_1 \times n_2 \times \ldots \times n_k$$

Example

- When designing a gearbox, we can choose to use the following different options.
  - 4 different fasteners
  - 3 different bolt lengths
  - 2 different bolt locations
- How many designs are possible?

Counting: Permutations Rule

- Finds the number of ordered sequences of the elements of a set.
- Consider a set of elements, such as $\mathcal{S} = \{a, b, c\}$. A **permutation** of the elements is an ordered sequence of the elements.
- For example, *abc, acb, bac, bca, cab,* and *cba* for all of the permutation of the elements of $\mathcal{S}$.
- The number of permutations of n different elements is $n!$ where:

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$$

- A permutation can be constructed by selecting the element to be placed in the first position of the sequence from the n elements, then selecting the element for the second position from the remaining $n-1$ elements, then selecting the element for the third position from the remaining $n-2$ element, and so forth.

Counting: Permutations Rule

- In some situations, we are interested in the number of arrangements of only some of the elements of a set.
- The number of permutations of subsets of r elements from a set of n different elements is:

$$P_r^n = n \times (n-1) \times (n-2) \times \ldots \times (n-r+1) = \frac{n!}{(n-r)!}$$

- A printed circuit board has eight different locations in which a component can be placed. If four different components are to be placed on the board, how may different designs are possible?

$$P_4^8 = \frac{8!}{4!}$$

Counting: Permutations Rule

- Sometimes we are interesting in counting the numbered of ordered sequences when **not all the items are different.**
- A hospital operating room needs to schedule 3 knee surgeries (denote as k) and 2 hip surgeries (denote as h) in a day. What are the number of possible sequences of 3 knee and 2 hip surgeries?
- $\{ \; kkkhh, \; kkhkh, \; kkhhk, \; \ldots, \; hhkkk \; \}$

$$\frac{5!}{2! \times 3!} = 10$$

- In general, the number of permutations of $n = n_1 + n_2 + \cdots + n_r$ objects of which n_1 are of one type, n_2 are of a second type, . . . , and n_r are of the r^{th} type is:

$$\frac{n!}{n_1! n_2! n_3! \ldots n_r!}$$

Counting: Combinations Rule

- Another counting problem is the number of subsets of r items that can be selected from a set of n where **order does not matter**.
- These are called **combinations**.
- A circuit board has eight locations in which a component can be placed. If 5 identical components are to be placed on a board, how many different designs are possible?
  - The order of the components is not important, so the combination rule is appropriate

$$\frac{3!}{2!1!} = 3 \quad \text{different ways}$$

Counting: Combinations Rule

- The number of combinations, subsets of r elements that can be selected from a set of n elements, is denoted as $\binom{n}{r}$ or C_r^n and

$$C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Counting: Combinations Rule

- For example: A bin of 50 manufactured parts contains 3 defective parts and 47 nondefective parts. A sample of 6 parts is selected from the 50 parts without replacement. How many different samples are there of size 6 that contain exactly 2 defective parts?

Example: Combinations Rule

- For example: A bin of 50 manufactured parts contains 3 defective parts and 47 nondefective parts. A sample of 6 parts is selected from the 50 parts without replacement. How many different samples are there of size 6 that contain exactly 2 defective parts?
  - The first step is to choose a subset containing exactly 2 defective parts by choosing 2 from the 3 defective parts.

$$C_2^3 = \binom{3}{2} = \frac{3!}{2!1!} = 3 \quad \text{different ways}$$

  - The second step is to select the remaining 4 parts from the 47 acceptable parts in the bin.

$$C_4^{47} = \binom{47}{4} = \frac{47!}{4!43!} = 178,365 \quad \text{different ways}$$

Counting: Combinations Rule

- Finally, we use the multiplication rule to get the number of subsets of size 6 that contain exactly 2 defective:

$$C_2^3 C_4^{47} = 3 \times 178,365 = 535,095$$

- Note that the total number of different subsets of size 6 can be found as:

$$C_6^{50} = \binom{50}{6} = \frac{50!}{6!44!} = 15,890,700 \text{ different ways}$$

- So the ratio of obtaining 2 defectives out 6 to any number (0-6) defectives out of 6 is:

$$\frac{C_2^3 C_4^{47}}{C_6^{50}} = 0.034$$

Example contd.

Example contd.

Sample Problem

- Plastic parts produced by an injection-molding operation are checked for conformance to specifications. Each tool contains 12 cavities in which parts are produced, and these parts fall into a conveyor when the press opens. An inspector chooses 3 parts from among the 12, at random. Two cavities are affected by a temperature malfunction that results in parts that do not conform to specifications.
    1. How many samples contain exactly 1 nonconforming part?
    2. How many samples contain at least 1 nonconforming part?

Sample Problem contd.

What is Probability?

- Probability is the likelihood or chance that a particular outcome or event from a random experiment will occur.
- Here, only finite sample spaces apply.
- Probability is a number in the $[0, 1]$ interval.
- May be expressed as a:
  - proportion (0.15)
  - percent (15%)
  - fraction (3/20)
- A probability of:
  - 1 means certainty
  - 0 means impossibility

Types of Probability

- Types of probability
  - Subjective probability is a **degree of belief**. "There is a 50% chance that he will study tonight."
  - Relative frequency probability is based on how often an event occurs over a very large sample space.
- Probabilities for a random experiment are often assigned on the basis of a reasonable model of the system under study.
- One approach is to base probability assignments on the simple concept of equally likely outcomes.

Types of Probability

- **Whenever a sample space consists of N possible outcomes that are equally likely, the probability of each outcome is $1/N$**
- Example:
  - In a batch of 100 diodes, 1 is colored red. A diode is randomly selected from the batch. Random means each diode has an equal chance of being selected. The probability of choosing the red diode is $1/100$ or 0.01, because each outcome in the sample space is equally likely.
- **The sum of probabilities must equal 1.**

Axioms of Probability

- Probability is a number that is assigned to each member of a collection of events from a random experiment that satisfies the following properties:
  1. $P(\mathcal{S}) = 1$
  2. $0 \leq P(E) \leq 1$
  3. For each two events E_1 and E_2 with $E_1 \cap E_2 = \emptyset$, $P(E_1 \cup E_2) = P(E_1) + P(E_2)$
- These imply that:
  1. $P(\emptyset) = 0$ and $P(E') = 1 - P(E)$
  2. If E_1 is contained in E_2, then $P(E_1) \leq P(E_2)$.

Probability Axioms

- Recall that $P(A \cap B) = P(A) + P(B) - P(A \cup B)$
- If events A and B are **mutually exclusive**:
  - $P(A \cap B) = \emptyset$
  - Therefore, $P(A \cup B) = P(A) + P(B)$
- Probabilities of joint events can often be determined from the probabilities of the individual events that comprise it. And conversely.

Example: Probability of Events

- A random experiment has a sample space $\{w, x, y, z\}$. These outcomes are not equally-likely; their probabilities are: $0.1, 0.3, 0.5, 0.1$. Recall, the sum of probabilities of the sample space is equal to 1.
- Event $A = \{w, x\}$, event $B = \{x, y, z\}$, event $C = \{z\}$
  - $P(A) = 0.1 + 0.3 = 0.4$
  - $P(B) = 0.3 + 0.5 + 0.1 = 0.9$
  - $P(C) = 0.1$
  - $P(A') = 0.6$ and $P(B') = 0.1$ and $P(C') = 0.9$
- Note that $A \cap B$, $A \cup B$, and $A \cup C$ are events that have probabilities.
  - Since event $A \cap B = \{x\}$, then $P(A \cap B) = 0.3$
  - Since event $A \cup B = \{w, x, y, z\}$, then $P(A \cup B) = 1.0$
  - Since event $A \cap C = \{\emptyset\}$, then $P(A \cap C) = 0.0$

Sample Problem

- A hospital operating room needs to schedule three knee surgeries and two hip surgeries in a day. Suppose that an operating room needs to schedule three knee, four hip, and five shoulder surgeries. Assume that all schedules are equally likely.
- Determine the probability for each of the following:
  1. All hip surgeries are completed before another type of surgery.
  2. The schedule begins with a hip surgery.
  3. The first and last surgeries are hip surgeries.
  4. The first two surgeries are hip surgeries.

Sample Problem contd.

More than Two Events

- The expression $P(A \cup B \cup C)$ can be expressed as follows:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) \\ - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A + B + C)$$

- **Derivation:**

Mutually Exclusive Events

- A collection of events E_1, E_2, E_k, is said to be mutually exclusive if for all pairs $E_i \cap E_j = \emptyset$.
- For a collection of mutually exclusive events:

$$P(E_1 \cup E_2 \cup \ldots \cup E_k) = P(E_1) + P(E_2) + \ldots + P(E_k)$$

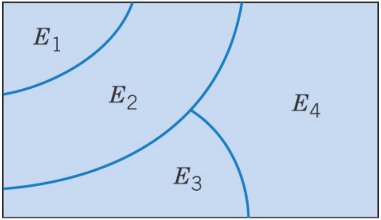


Figure: Below is a Venn diagram of four mutually exclusive events. Note that no outcomes are common to more than one event, i.e. all intersections are null.

Sample Problem

- A clinical study of Rheumatoid Arthritis considered four treatment groups. The groups consisted of patients with different drug therapies: sequential monotherapy (group 1), step-up combination therapy (group 2), initial combination therapy (group 3), or initial combination therapy with infliximab (group 4). Radiographs of hands and feet were used to evaluate disease progression. The number of patients without progression of joint damage was 76 of 114 patients (67%), 82 of 112 patients (73%), 104 of 120 patients (87%), and 113 of 121 patients (93%) in groups 1–4, respectively.
- Suppose that a patient is selected randomly. Let A denote the event that the patient is in group 1, and let B denote the event that there is no progression. Determine the following probabilities:
  1. $P(A \cup B)$
  2. $P(A' \cup B')$
  3. $P(A \cup B')$

Sample Problem contd.

Sample Problem contd.

Conditional Probability

- Sometimes probabilities need to be reevaluated as additional information becomes available.
- $P(B|A)$ is called the **conditional probability**, i.e., probability of event B occurring, given that event A has already occurred.
- The conditional probability of event B given event A, can also be expressed as follows

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \quad \text{for} P(A) > 0$$

Manufacturing Example

- In a thin film manufacturing process, a proportion of parts produced are generally not acceptable. However, the process is sensitive to contamination, which increases the rate of part rejections. Contamination often produces surface flaws.
- The Table below provides the results from a sample of 400 parts classified by surface defects and as (functionally) detective.
- Let D denote the event that a part is defective, and let F denote the event that a part has a surface flaw.

Parts Classified

	Surface Flaws		
Defective	Yes (F)	No (F')	Total
Yes (D)	10	18	28
No (D')	30	342	372
Total	40	360	400

Manufacturing Example contd.

Parts Classified			
	Surface Flaws		
Defective	Yes (F)	No (F')	Total
Yes (D)	10	18	28
No (D')	30	342	372
Total	40	360	400

- Find the probability that the part is defective given that it has a surface flaw?

$$P(D|F) = 10/40 = 0.25$$

- Find the probability that the parts without a surface flaw are defective.

$$P(D|F') = 18/360 = 0.05$$

- **Practical interpretation:** The probability of being defective is five times greater for parts with surface flaws. This calculation illustrates how probabilities are adjusted for additional information.

Sample Problem

A batch of 500 containers for frozen orange juice contains 5 that are defective. **Two containers are selected at random without replacement from the batch**.

1. What is the probability that the second one selected is defective given that the first one was defective?
2. What is the probability that both are defective?
3. What is the probability that both are acceptable?

Sample Problem

A batch of 500 containers for frozen orange juice contains 5 that are defective. **Three containers are selected at random without replacement from the batch.**

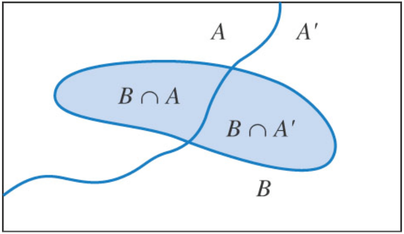
1. What is the probability that the third one selected is defective given that the first and second ones selected were defective?

2. What is the probability that the third one selected is defective given that the first one was defective and the second one selected was okay?

3. What is the probability that all three are defective?

Total Probability Rule

- Sometimes the probability of an event is given under each of several conditions. With enough of these conditional probabilities, the probability of the event can be recovered.
- To see this, consider any event B. We can write B as the union of the part of B in A and the part of B in A'.
- That is $B = (A \cap B) \cup (A' \cap B)$. The result is the Venn diagram shown below.



Total Probability Rule

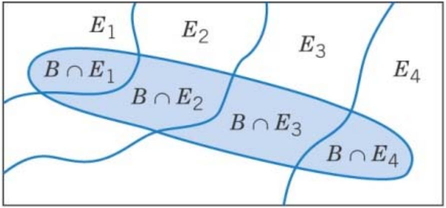
- We have $B = (A \cap B) \cup (A' \cap B)$.
- Because A and A' are mutually exclusive, then $(A \cap B)$ and $(A' \cap B)$ will also be mutually exclusive.
- Therefore from the probability of the union of mutually exclusive events and the multiplication rule that we saw earlier, we get the following **total probability rule** for any events A and B:

$$P(B) = P(A \cap B) + P(A' \cap B) = P(B|A)P(A) + P(B|A')P(A')$$

Total Probability Rule (multiple events)

- Assume $E_1, E_2, \ldots E_k$ are k mutually exclusive & exhaustive subsets. Then:

$$\begin{aligned} P(B) &= P(B \cap E_1) + P(B \cap E_2) + \ldots + P(B \cap E_k) \\ &= P(B|E_1)P(E_1) + P(B|E_2)P(E_2) + \ldots + P(B|E_k)P(E_k) \end{aligned}$$



$$B = (B \cap E_1) \cup (B \cap E_2) \cup (B \cap E_3) \cup (B \cap E_4)$$

Example

- In a semiconductor manufacturing, assume the following probabilities for product failure subject to levels of contamination in the manufacturing process.

Probability of Failure	Level of Contamination
0.10	High
0.01	Medium
0.001	Low

- In a particular production run 20% of the chips are subjected to high levels of contamination, 30% to medium levels of contamination, and 50% to low levels of contamination. What is the probability that a product using one of these chips fails?
  1. **H** denote the event that a chip is exposed to high levels of contamination.
  2. **M** denote the even that a chip is exposed to medium levels of contamination.
  3. **L** denote the event that a chip is exposed to low levels of contamination.

Example contd.

Example contd.

Sample Problem

A lot of 100 semiconductor chips contains 20 that are defective.

- Two are selected, at random, without replacement, from the lot. Determine the probability that the second chip selected is defective.
- Three are selected, at random, without replacement, from the lot. Determine the probability that all are defective

Another Sample Problem

A hospital operating room needs to schedule three knee surgeries and two hip surgeries in a day. Suppose that an operating room needs to schedule three knee, four hip, and five shoulder surgeries. Assume that all schedules are equally likely. **Determine the following probabilities**:

1. All hip surgeries are completed first given that all knee surgeries are last.
2. The schedule begins with a hip surgery given that all knee surgeries are last.
3. The first and last surgeries are hip surgeries given that knee surgeries are scheduled in time periods 2 through 4.
4. The first two surgeries are hip surgeries given that all knee surgeries are last.

Problem contd.

Problem contd.

Problem contd.

Independence

- Two events are independent if any one of the following equivalent statements is true:
  - $P(A|B) = P(A)$
  - $P(B|A) = P(B)$
  - $P(A \cap B) = P(A) \cdot P(B)$
- This means that occurrence of one event has no impact on the probability of occurrence of the other event.

Baye's Theorem

- Baye's theorem provides a useful result that allows us to solve $P(A|B)$ in terms of $P(B|A)$.

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

- The denominator can also be expanded in terms of condition probabilities.
- If $E_1, E_2, \ldots E_k$ are k mutually exclusive and exhaustive events and B is any event,

$$P(E_1|B) = \frac{P(B|E_1) \cdot P(E_1)}{P(B|E_1)P(E_1) + P(B|E_2)P(E_2) + \ldots + P(B|E_k)P(E_k)}$$

Problem

- Suppose that $P(A|B) = 0.7$, $P(A) = 0.5$, and $P(B) = 0.2$. Determine $P(B|A)$.

Problem

- Suppose that $P(A|B) = 0.4$, $P(A|B') = 0.2$, and $P(B) = 0.8$. Determine $P(B|A)$.

Random Variables

- A variable that associates a number with the outcome of a random experiment is called a random variable.
- A random variable is a function that assigns a real number to each outcome in the sample space of a random experiment.
- A random variable is denoted by an uppercase letter such as X. After the experiment is conducted, the measured value of the random variable is denoted by a lowercase letter such as $x = 70$ milliamperes. X and x are shown in italics, e.g., $P(X = x)$.

Types of Random Variables

- A discrete random variable is a random variable with a finite or countably infinite range. Its values are obtained by counting.
  - Number of scratches on a surface.
  - Proportion of defective parts among 100 tested.
- A continuous random variable is a random variable with an interval (either finite or infinite) of real numbers for its range. Its values are obtained by measuring.
  - Electrical current and voltage.
  - Physical measurements, e.g., length, weight, time, temperature, pressure.